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Abstract: This paper focuses on asymptotic point-stabilization of smooth affine control systems. For asymptotic feedback stabilizers, a simple definition of *robustness* with respect to unmodeled dynamics is adopted. Two theorems are then proved which state sufficient conditions for the *non-robustness* of homogeneous stabilizers. The first theorem, which applies to systems that may contain a drift term, involves a specific class of feedback stabilizers. The second one, which applies to driftless systems and is stated independently of any particular stabilizer, provides a condition related to an eventual loss of rank of the accessibility distribution. One of the consequences of the second result is that, for *chained-form systems*, no (static) continuous homogeneous exponential stabilizer (some of which have been proposed in the literature) can be robust in the sense defined herein. Examples are provided which illustrate a typical application of each result.

Key-words: Robust stabilization, homogeneous feedback, nonholonomic systems.

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Non-robustesse des stabilisateurs continus, homogènes pour des systèmes de commande affines

Résumé : Nous considérons la stabilisation asymptotique des systèmes différentiables affines en la commande. Partant d'une définition de la robustesse par rapport à des dynamiques non modélisées, nous démontrons deux théorèmes qui énoncent des conditions suffisantes pour la *non-robustesse* des stabilisateurs continus homogènes. Le premier résultat porte sur des systèmes comportant éventuellement un terme de dérive et a trait à une classe spécifique de stabilisateurs. Le second porte sur des systèmes sans dérive et est énoncé indépendamment du stabilisateur utilisé; il établit une condition de non-robustesse rapportée à une éventuelle perte de rang de la distribution d'accessibilité. Une conséquence de ce dernier résultat est que, pour les systèmes sous *forme chaînée*, aucun retour d'état (statique) continu, homogène, et rendant l'origine exponentiellement stable n'est robuste dans le sens considéré. Ces deux théorèmes sont illustrés à l'aide d'exemples.

Mots-clés : Stabilisation robuste, retour d'état homogène, systèmes non-holonomes.

1 Introduction

Homogeneity has played an important role in the development of nonlinear control theory, particularly during the last two decades, and two major reasons account for its importance. In the first place, homogeneous approximations enable the investigation of local properties of nonlinear systems through the study of "better behaved" vector fields; typically, the approximating vector fields have polynomial components and span nilpotent and/or finitely generated Lie algebras, a fact which results in simpler computations. Effective methods exist for the construction of homogeneous approximations; see Stefani [34, 35] and Hermes [10], for instance. As pointed out in those references, homogeneous approximations can be so constructed as to preserve basic properties of the original system, local controllability being one of them. The second reason is the usefulness of homogeneity in solving the feedback stabilization problem for nonlinear systems. Indeed, homogeneity can be visualized as a natural extension of linearity, and several properties of linear systems are also valid for homogeneous ones via minor adjustments. The cornerstone result in this direction, proved by Rosier in [28], basically states that if the continuous vector field f is homogeneous and the origin of $\dot{x} = f(x)$ is asymptotically stable, then there exists a sufficiently differentiable *homogeneous* Lyapunov function. A useful corollary, derived by Pomet and Samson in [27], extends Rosier's result to the case where f depends periodically on time. One domain in which these properties are fruitful is the synthesis of feedback stabilizers for *nonholonomic systems* (in our context, these are driftless systems with less control inputs than state variables). As is presently well known, such systems do not satisfy Brockett's necessary condition for *continuous* pure state feedback stabilization (cf. Brockett [3], Sontag [32]). To elude the use of discontinuous stabilizers, one effective approach consists in using smooth *time-varying* feedback laws. This technique was successfully introduced by Samson in [29] to asymptotically stabilize configurations (i.e. points in the state-space) of a cart-like, nonholonomic mobile robot. Soon after, Coron [4] showed that any smooth driftless system can be stabilized to a point by smooth time-periodic feedback. Constructive approaches were proposed by Pomet [26] for a class of driftless systems, and by Coron and Pomet [6] for controllable driftless systems. However, an important shortcoming of smooth time-periodic stabilizers is that, for most initial conditions, the rate of convergence is not exponential, as was first pointed out in Samson and Ait-Abderrahim [31]. Murray et al. [23] showed that smooth, time-periodic feedback cannot exponentially stabilize nonholonomic systems to a point, whereas Gurvits and Li [8] proved that Hölder continuity of the control laws was the highest regularity one could aim at in order to achieve exponential convergence. M'Closkey and Murray carefully studied the convergence rate of solutions, and in [15, 16] they proposed a Hölder continuous stabilizer for low-dimensional systems in power-form. Their feedback (non-Lipschitz at the origin but smooth elsewhere) rendered the closed-loop system homogeneous of degree zero and the origin was shown to be locally ρ-exponentially stable. These results spurred further development of continuous non-smooth homogeneous stabilizers and, to date, there exist several feedback controls which follow related principles and ensure ρ-exponential stability (cf. for instance [17, 27, 18, 19, 20]).

One question which then arises is: *Do these continuous (time-varying) homogeneous feedbacks preserve ρ-exponential stability when the actual system's vector fields differ slightly from those used to design the feedback law?* The present paper addresses this issue by first proposing a definition of robustness of a feedback stabilizer with respect to *unmodeled dynamics*, and then proving two theorems. Roughly stated, these theorems establish that for some classes of systems with non-controllable linear approximations, robust exponential stabilization cannot be achieved by means of continuous homogeneous feedback. As we shall see, our results concern systems with drift (e.g. the under-actuated satellite) as well as systems without drift (e.g. kinematic models of mobile robots).

The outline of the paper is as follows. In Section 2 we recall basic concepts related to homogeneity. In Section 3 we define what will be meant by robustness with respect to a class of model perturbations. Section 4 contains the main results and their proofs. In Section 5 we discuss two examples on how these results are typically applied. Section 6 provides some concluding remarks.

To simplify the notation we let $\mathbb{R}^{*n} := \mathbb{R}^n \backslash \{0\}$ and make use of Bachmann-Landau's notation as follows. Given functions $f \in C^0(\mathbb{R}^{*n} \times \mathbb{R}; \mathbb{R})$ and $g \in C^0(\mathbb{R}^n; \mathbb{R})$ such that $g(x) = 0$ iff $x = 0$, we write "f is $o(g)$" (or $f = o(g)$) if for all $t \in \mathbb{R}$

$$\lim_{x \to 0} \frac{|f(x,t)|}{|g(x)|} = 0.$$

Similarly, "f is $O(g)$" (or $f = O(g)$) means that for any $t \in \mathbb{R}$ there exist a real number $K > 0$ and a neighborhood B of the origin such that

$$\frac{|f(x,t)|}{|g(x)|} \leq K$$

for every $x \in B \backslash \{0\}$.

2 Preliminary Recalls

The forthcoming recalls are limited to the basic definitions and properties of homogeneous systems. For more detailed discussions, the reader is referred to e.g. Hermes [10], Kawski [11, 12] or Olver [25].

2.1 Homogeneous Mappings

An n-tuple of strictly positive reals $r = (r_1, \ldots, r_n)$ is called a **weight vector**. Given a weight vector r and a real $\lambda > 0$, the mapping $\Delta_\lambda^r : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Delta_\lambda^r(x) = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n)^T$$

is called a **dilation of weight** r. The same term is often used to refer to *the whole family* of dilations indexed by $\lambda > 0$. A dilation of weight r will be denoted as Δ^r. Observe that when $r = (1, 1, \dots, 1)$, one recovers the classical concept of homogeneity, the dilations become homotheties, and in this case the dilation is referred to as the **standard dilation** Δ^{std}.

Consider a dilation Δ^r. A function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is said to be **homogeneous of degree** τ **with respect to** Δ^r (or briefly, r-homogeneous of degree τ) if $f(\Delta^r_\lambda(x,t)) = \lambda^\tau f(x,t)$ for any $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. A vector field $h : (x,t) \mapsto h(x,t) = (h_1(x,t), \dots, h_n(x,t))^T$ is said to be **homogeneous of degree** σ **with respect to** Δ^r (or briefly, r-homogeneous of degree σ) if, for each $i = 1, \dots, n$, the function h_i is r-homogeneous of degree $\sigma + r_i$. A **homogeneous norm associated with a dilation** Δ^r (or briefly, an r-norm) is a continuous positive-definite mapping $\rho : \mathbb{R}^n \to \mathbb{R}$ which is r-homogeneous of degree one.

Note that for $k > 0$, a function f which is r-homogeneous of degree τ is kr-homogeneous of degree $k\tau$; this implies that the weights can always be normalized so as to have $r_1 = 1$. Also, by eventually renaming the coordinates, we can arrange the components of r in nondecreasing order. Thus, without loss of generality, we assume throughout that $1 = r_1 \leq r_2 \leq \cdots \leq r_n$.

Assume the components of the weight vector $r = (r_1, \dots, r_n)$ take $N \geq 1$ distinct values, denoted as $r^1, \dots, r^N$ with $1 = r^1 < \cdots < r^N = r_n$. For any $\mathbb{R}^n$-valued map f, $\mathbf{f}$ will represent the r**-partition of** f, which consists of N sub-vectors $\mathbf{f}_1, \dots, \mathbf{f}_N$ such that $f = (\mathbf{f}_1, \dots, \mathbf{f}_N)^T$, and such that f_k ($k \in \{1, \dots, n\}$) is an entry of $\mathbf{f}_j$ ($j \in \{1, \dots, N\}$) iff $r_k = r^j$. For instance, if $r = (1,3,3,3)$ and $f = (f_1, f_2, f_3, f_4)^T$, then $N = 2$, $r^1 = 1, r^2 = 3$, and $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2)^T$ with $\mathbf{f}_1 = (f_1)$, $\mathbf{f}_2 = (f_2, f_3, f_4)$. The use of bold symbols (e.g. $\mathbf{f}, \mathbf{x}$) to denote partitions of vectors will be consistently conserved hereafter.

2.2 Some Properties of Homogeneous Vector Fields

Let Δ^r be a dilation and $X : x \mapsto X(x)$ a smooth r-homogeneous vector field of degree δ. Let $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_N)$ and $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N)$ denote the r-partitions of X and x, respectively. Then:

P1. *Each component of X is a polynomial function.*

P2. *If the smooth vector field X' is r-homogeneous of degree δ', then $[X, X']$ (the Lie bracket) is r-homogeneous of degree $\delta + \delta'$.*

P3. *For each $j \in \{1, \dots, N\}$, the components of $\mathbf{X}_j$ are:*

a) *identically zero if $r_j < -\delta$,*
b) *constant if $r_j = -\delta$,*
c) *functions that vanish at the origin if $r_j > -\delta$.*

P4. *Unless $X \equiv 0$, its degree δ has a lower bound: $\delta \geq -r^N$.*

P5. *If $r = (1, \dots, 1)$ and $\delta < 0$, then X is constant.*

P6. *If $\delta < 0$ then, for each $j \in \{1, \dots, N\}$, only $\mathbf{x}_1, \dots, \mathbf{x}_{j-1}$ may appear explicitly in $\mathbf{X}_j$.*

2.3 Homogeneous Approximations

Let

$$\dot{z} = f_0(z) + \sum_{k=1}^{m} f_k(z) u_k \tag{1}$$

represent an affine control system defined in $\mathbb{R}^n$, with local coordinates $z = (z_1, \dots, z_n)^T$, control vector $u = (u_1, \dots, u_m)^T \in \mathbb{R}^m$ and $f_0, \dots, f_m$ smooth vector fields. Unless otherwise stated, we assume that $f_0(0) = 0$. For simplicity of exposition, we shall slightly abuse notation by sometimes calling *vector field* an application $f : \mathbb{R}^n \times \mathbb{R} \to T\mathbb{R}^n$ mapping (z, t) to $f(z, t)$ instead of viewing it as a family of vector fields indexed by t.

In our framework, an **r-homogeneous approximation of system (1) with coordinate change $x = \varphi(z)$ and weight vector r** is a system

$$\dot{x} = b_0(x) + \sum_{k=1}^{m} b_k(x) u_k, \tag{2}$$

with $b_0, \dots, b_m$ smooth vector fields, such that:

- b_0 is r-homogeneous of degree 0, and for $k = 1, \dots, m$, b_k is r-homogeneous of degree $\delta_k < 0$;
- in x-coordinates the system (1) is given by

$$\dot{x} = b_0(x) + h_0(x) + \sum_{k=1}^{m} (b_k(x) + h_k(x)) u_k, \tag{3}$$

  with $h_{j,i} = o(\rho^{r_i + \delta_j})$ $(j = 0, \dots, m,\ i = 1, \dots, n)$ for any r-norm ρ ($h_{j,i}$ is the i-th component of the vector field h_j);
- the Lie Algebra Rank Condition (LARC) holds at the origin:

$$\operatorname{span}\{X(0) : X \in \operatorname{Lie}(b_0, \dots, b_m)\} = \mathbb{R}^n. \tag{4}$$

Remark This definition applies equally well to driftless systems (in which case $f_0 = b_0 = 0$). For (symmetric) driftless systems, (4) is a sufficient condition for local controllability at the origin. For systems with drift term, however, this definition is not very demanding in the sense that (4) is weaker (and easier to evaluate) than Sussmann's sufficient condition for Small-Time Local Controllability (STLC).

3 Problem Statement

In linear systems theory (see e.g. Feintuch [7] and the references therein), robustness of a feedback law (with respect to unmodeled dynamics) may be defined in terms of a space of operators endowed with a suitable topology or metric. A system is represented by *a point* in that space, and a feedback law which stabilizes a nominal system is called robust if it stabilizes every system located *sufficiently near* (i.e. in a small neighborhood of) the nominal system. Whereas a similar general approach can also be used in the case of nonlinear systems, we adopt a narrower perspective by considering a nominal system subject to perturbations which, basically, are additive terms with sufficient regularity. Essentially, this viewpoint was also adopted in the definition of robustness given in Bennani and Rouchon [2].

Definition 1 *An* **admissible perturbation** *is a smooth mapping $\varepsilon \mapsto g(\varepsilon, \cdot)$ such that for every $\varepsilon \in \mathbb{R}$, $g(\varepsilon, \cdot)$ is a smooth vector field in $\mathbb{R}^n$ and $g(0,x) = 0$ for all $x \in \mathbb{R}^n$.*

Definition 2 *Let $\alpha : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ be a feedback law such that the origin of the "nominal" system (1), with $u = \alpha(z,t)$, is locally asymptotically stable. The feedback α will be called* **robust** *if, given any family of admissible perturbations $g_0, \dots, g_m$ satisfying $g_0(\varepsilon, 0) = 0$ for every $\varepsilon \in \mathbb{R}$, there exists $\varepsilon_0 > 0$ such that the origin of the "perturbed" system*

$$\dot{z} = f_0(z) + g_0(\varepsilon, z) + \sum_{k=1}^{m} (f_k(z) + g_k(\varepsilon, z))\alpha_k(z,t) \tag{5}$$

is locally asymptotically stable for every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.

As defined above, admissible perturbations may be viewed as vector fields whose perturbing effect can be made as "small" as desired by the choice of a small enough ε. If ε quantifies the "uncertainty" in the knowledge of the real system with respect to the nominal one, the foregoing definitions are reasonable to model various sources of error in real control systems. Nonetheless, additive perturbations —including more general classes, such as discontinuous ones— are far from being exhaustive to model all possible errors (parameter errors, state measurement errors, timing errors, etc.).

Concerning the feedback laws, we shall focus our attention on *continuous homogeneous ρ-exponential* stabilizers. Prior to giving a precise definition of such feedbacks, let us recall some notions related to stability. Consider a continuous system

$$(S): \quad \dot{x} = f(x,t)$$

for which the origin $x = 0$ is an equilibrium. The origin is said to be **stable** (in the sense of Lyapunov) if for any $t_0 \in \mathbb{R}$ and any $\eta > 0$, there exists $\delta > 0$ such that

$$\|x_0\| < \delta \quad \Longrightarrow \quad \|x(t,t_0,x_0)\| < \eta, \quad \forall t \geq t_0,$$

where $x(\cdot, t_0, x_0)$ represents a solution to (S) issued from x_0 at $t = t_0$. The origin is **unstable** if for some reals t_0 and $\eta > 0$, and some sequence $(x_{0,k})_{k\in\mathbb{N}}$ which converges to zero, there exists a sequence $(t_k)_{k\in\mathbb{N}}$ such that $\|x(t_k, t_0, x_{0,k})\| \geq \eta$ (with $x_{0,k} \in \mathbb{R}^n$, $t_k \geq t_0$, $\forall k \in \mathbb{N}$). Finally, given a dilation Δ^r, the origin is is said to be (locally) **ρ-exponentially stable** if, for any r-norm ρ, there exist real numbers $\delta > 0$, $K > 0$ and $\gamma > 0$ such that for every $t_0 \geq 0$

$$\rho(x_0) < \delta \quad \Longrightarrow \quad \rho(x(t,t_0,x_0)) \leq K\rho(x_0)e^{-\gamma(t-t_0)}, \quad \forall t \geq t_0.$$

Definition 3 *Suppose that (2) is an r-homogeneous approximation of system (1), with coordinate change $x = \varphi(z)$. The feedback law $\alpha = (\alpha_1, \ldots, \alpha_m)^T \in C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ will be called a* **continuous homogeneous ρ-exponential stabilizer for (1)** *if*

(i) α_k is r-homogeneous of degree $-\delta_k$ $(k = 1, \ldots, m)$,

(ii) the origin of the approximating system (2), with $u = \alpha(x, t)$, is ρ-exponentially stable.

Observe that, when expressed in the original coordinates $z = \varphi^{-1}(x)$, a feedback α meeting the requirements of the last definition will also (locally) ρ-exponentially stabilize the origin of (1). Indeed, the existence of an r-homogeneous Lyapunov function is guaranteed, in that case, by a trivial extension of Proposition 4 in Pomet and Samson [27]. Therefore, the local ρ-exponential stability ensured by α is preserved after the addition of perturbing terms which, in closed-loop, result in sums of r-homogeneous vector fields of strictly positive degrees. In that sense α is "robust" to the addition of such perturbations. Nevertheless, α may fail to be robust in the sense of Definition 2, and the main contribution of the present paper consists precisely in pointing out conditions under which α *is not* robust.

4 Main Results

4.1 Statement

We first state Theorems 1 and 2, the main results in the paper, and make some remarks before proceeding to their proofs.

Theorem 1 *Assume that (1) has an r-homogeneous approximation (2) with coordinate change $x = \varphi(z)$. Let α be a continuous homogeneous ρ-exponential stabilizer for (1) (cf. Definition 3). Suppose that for some $t_0 \in \mathbb{R}$ and some integers $p \in \{1, \ldots, m\}$ and $q \in \{1, \ldots, n\}$*

$$\alpha_p(\mathrm{e}_q, t_0) \neq 0 \quad \textit{and} \quad -\delta_p < r_q, \tag{6}$$

where $\mathrm{e}_q \in \mathbb{R}^n$ has 1 at the q-th entry and 0's elsewhere. Then, for any $\varepsilon > 0$, the family of admissible perturbations g_k $(k = 0, 1, \ldots, m)$ defined by

$$g_k(\varepsilon, x) := \begin{cases} \varepsilon \operatorname{sgn}(\alpha_p(\mathrm{e}_q, t_0))\mathrm{e}_q, & \textit{if } k = p, \\ 0, & \textit{otherwise}, \end{cases} \tag{7}$$

makes the origin of the perturbed closed-loop system, which in x-coordinates is

$$\dot{x} = b_0(x) + h_0(x) + g_0(x) + \sum_{k=1}^{m} (b_k(x) + h_k(x) + g_k(\varepsilon, x))\alpha(x, t), \tag{8}$$

unstable. Therefore, α is not robust in the sense of Definition 2.

The next result follows from Theorem 1 in the specific case of driftless systems.

Theorem 2 *Consider a system of the form (1) with $f_0 = 0$ (i.e. a driftless system):*

$$\dot{z} = \sum_{k=1}^{m} f_k(z) u_k. \tag{9}$$

Suppose that (9) satisfies the LARC (4) and assume that for every $p \in \{1, \ldots, m\}$

$$\dim \operatorname{span}\{X(0) : X \in \operatorname{Lie}(f_1, \ldots, f_{p-1}, f_{p+1}, \ldots, f_m)\} < n - 1 \tag{10}$$

(i.e., by removing f_p the accessibility distribution at zero loses more than one dimension). Then, no continuous homogeneous ρ-exponential stabilizer for (9) is robust in the sense of Definition 2.

Remarks

1) Observe that the evaluation of the "rank-loss" condition (10) does not require the computation of any homogeneous approximation. This is convenient since the latter is a process which often requires significant effort.

2) As to Theorem 2, it is well known that when the linear approximation $\dot{x} = \sum_{k=1}^{m} b_k(0) u_k$ of the nominal (driftless) system is controllable, one can find linear feedback controls that ensure local, asymptotic stability of the origin. Moreover, it is easy to check that any such control is robust in the sense defined above. For our results to be of value, the assumptions must therefore exclude this class of systems. This is done via the requirement (10), which implies that the linear approximation is not controllable. Indeed, if it were controllable, $\operatorname{rank}(b_1(0), \ldots, b_m(0))$ would be n and, by removing any vector field b_p, the accessibility distribution at zero would lose at most one dimension.

4.2 Proofs

A key element in the proof of our first result is Proposition 1 below, which may be viewed as a modified version of one of the theorems usually referred to as *Lyapunov-Chetaev's instability theorems* (see e.g. Hahn [9, Sect. 25], Vidyasagar [36] or Khalil [13]). In order to simplify the forthcoming discussion, we use the following notational conventions. Given a real $\kappa > 0$ and a point $p \in \mathbb{R}^n$, we set

$$B_\kappa(p) := \{x \in \mathbb{R}^n : \|x - p\| \leq \kappa\}$$

(i.e. $B_\kappa(p)$ is the closed ball of radius κ centered at p). Also, given a set $S \subset \mathbb{R}^n$, we will denote its interior as $\operatorname{Int}(S)$ and its boundary as ∂S.

Proposition 1 *Consider the system*

$$\dot{x} = f(x, t), \tag{11}$$

where $f \in C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R})$ and $f(0, t) = 0$ for every $t \in \mathbb{R}$. Assume that there exist a function $V \in C^1(\mathbb{R}^n; \mathbb{R})$ and an open set $A \subset \mathbb{R}^n$, with $0 \in \partial A$, such that $V(x) > 0$ if $x \in A$ and $V(x) = 0$ if $x \in \partial A$. Assume, moreover, that there exist

- *an open interval $\mathcal{T} \subset \mathbb{R}$,*
- *real constants $K > 0$, $\gamma \in]0,1[$ and $\kappa > 0$,*

such that:

$$\forall (x,t) \in [A \cap B_\kappa(0)] \times \mathcal{T} : \quad (L_f V)(x,t) \geq K[V(x)]^\gamma . \tag{12}$$

Then the origin $x = 0$ of (11) is unstable.

Proof. Let $t_0 \in \mathcal{T}$ and choose $\tau > 0$ such that $\mathcal{T}' := [t_0, t_0 + \tau] \subset \mathcal{T}$. We proceed by contradiction and assume that the origin of (11) is stable. This assumption implies that for any $x_0 \in A \cap B_\kappa(0)$ with $\|x_0\|$ sufficiently small, the solution $x(t)$ to (11), issued from x_0 at $t = t_0$, is defined for $t \in [t_0, \infty[$ and satisfies $x(t) \in B_\kappa(0)$ for every $t \geq t_0$. Let $\overline{V}(t) := V(x(t))$ so that $\overline{V}(t_0) = V(x_0) > 0$ and $(d\overline{V}/dt)(t) = (L_f V)(x(t), t)$. We claim that $x(t)$ cannot reach ∂A for any $t \in \mathcal{T}'$. Indeed, assume $t_1 \in \mathcal{T}'$ is such that $x(t_1) \in \partial A$ and $x(t) \in A \cap B_\kappa(0)$ for every $t \in [t_0, t_1[$. Hence $\overline{V}(t_1) = 0 < \overline{V}(t_0)$, which means that $\overline{V}$ decreases while $(x(t), t) \in [A \cap B_\kappa(0)] \times \mathcal{T}$. But this is a contradiction to (12), so the claim follows. Therefore, the solution $x(t)$ remains in $A \cap B_\kappa(0)$ for all $t \in \mathcal{T}'$. From (12), we see that $(d\overline{V}/dt)(t) \geq K[\overline{V}(t)]^\gamma$ and, by direct integration,

$$\overline{V}(t) \geq \left[K(1-\gamma)(t-t_0) + [\overline{V}(t_0)]^{1-\gamma}\right]^{\frac{1}{1-\gamma}} .$$

By evaluating at $t = t_0 + \tau$ and using $[\overline{V}(t_0)]^{1-\gamma} > 0$, we get

$$\overline{V}(t_0 + \tau) > [K(1-\gamma)\tau]^{\frac{1}{1-\gamma}} > 0. \tag{13}$$

The continuity of $\overline{V}$, the assumption $V(0) = 0$, and the inequality (13) imply that the set $C = \{x \in A : V(x) = [K(1-\gamma)\tau]^{\frac{1}{1-\gamma}}\}$ is non-empty and closed, so one easily checks (again by continuity of $\overline{V}$) that

$$d := \inf_{x \in C} \|x\|$$

exists, and that $d > 0$. Note that d does not depend on the choice of x_0. From what precedes, one concludes that the origin is unstable since the norm of the solution invariably attains the strictly positive value d. This contradicts the assumption that the origin is stable. ∎

Remark Instead of assumption (12), one of Lyapunov-Chetaev's theorems requires that there exist a class $\mathcal{K}$ function h such that $(L_f V)(x,t) \geq h(\|x\|)$ for all $(x,t) \in [A \cap B_\kappa(0)] \times \mathbb{R}$. However, under the assumptions of Theorem 1 this need not be the case since the closed-loop system depends on the time-varying feedback α, and the latter may vanish independently of the magnitude of $\|x\|$. The proof of Proposition 1 only requires the existence of an open time-interval on which $(L_f V)(\cdot, t)$ is not Lipschitz at the origin.

4.2.1 Proof of Theorem 1

In x-coordinates, the perturbed closed-loop system has the form

$$\dot{x} = \tilde{f}(x,t) + g_0(\varepsilon, x) + \sum_{k=1}^{m} g_k(\varepsilon, x)\alpha_k(x,t), \tag{14}$$

with

$$\tilde{f}(x,t) := b_0(x) + h_0(x) + \sum_{k=1}^{m}(b_k(x) + h_k(x))\alpha_k(x,t)$$

denoting the "unperturbed" closed-loop vector field. Since the right-hand member of (14) is continuous (but not necessarily Lipschitz), the existence of solutions is guaranteed for a sufficiently small non-empty time-interval $[t_0, t_1[$. Uniqueness of solutions, which in this case need not hold, is not required to prove instability of the origin. Also, by using the homogeneity assumptions, we see that each component $\tilde{f}_i$ of $\tilde{f}$ is $O(\rho^{r_i})$ for any r-norm ρ. Our aim is to prove that admissible perturbations $g_0, \ldots, g_m$, as defined in (7), can render the origin of (14) unstable. This will be accomplished by showing that (14) verifies the assumptions of Proposition 1 for any choice of $\varepsilon > 0$. To this purpose, we will construct a function $V : \mathbb{R}^n \to \mathbb{R}$ and a set A as required in the proposition. Let

$$\rho(x) = \left(\sum_{i=1}^{n} x_i^{\frac{2c}{r_i}}\right)^{\frac{1}{2c}}, \quad \text{with} \quad c = \prod_{i=1}^{n} r_i, \tag{15}$$

and define the auxiliary mapping $\psi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\begin{aligned} \psi(x) &= \rho(x)\Delta_{\frac{1}{\rho(x)}}(x) \\ &= (\rho^{1-r_1}(x)x_1, \ldots, \rho^{1-r_n}(x)x_n)^T \quad (x \neq 0), \end{aligned}$$

and $\psi(0) = 0$. It is simple to show that ψ is a homeomorphism and that it is continuously differentiable in $\mathbb{R}^{*n}$. For the forthcoming computations, it is convenient to introduce the change of coordinates $y = \psi(x)$. For a function $f : \mathbb{R}^{*n} \to \mathbb{R}$ (respectively a vector field $X : \mathbb{R}^{*n} \times \mathbb{R} \to T\mathbb{R}^n$) given in x-coordinates, we will denote its expression in y-coordinates as $\psi_* f$ (respectively $\psi_* X$). Note that if f is r-homogeneous of degree d, then $\psi_* f = f \circ \psi^{-1}$ is homogeneous of same degree d, but with respect to the *standard* dilation Δ^{std}. Also, if $f = o(\rho^\tau)$ (respectively $f = O(\rho^\tau)$) then $\psi_* f = o(\|\cdot\|^\tau)$ (respectively $\psi_* f = O(\|\cdot\|^\tau)$). Similarly, if a continuous vector field X is r-homogeneous of degree d, then $\psi_* X$ is Δ^{std}-homogeneous of degree d. If the vector field $X = (X_1, \ldots, X_n)^T$ is continuous and satisfies $X_i = o(\rho^{d+r_i})$ (respectively $X_i = O(\rho^{d+r_i})$) for $i = 1, \ldots, n$, then $\psi_* X = (\psi_* X_1, \ldots, \psi_* X_n)$ satisfies $\psi_* X_i = o(\|\cdot\|^{d+1})$ (respectively $\psi_* X_i = O(\|\cdot\|^{d+1})$). The verification of these assertions represents no particular difficulty. In the sequel, we will simply write "homogeneous" instead of "homogeneous with respect to Δ^{std}".

Any solution of (14), with initial condition $x(t_0) \neq 0$, satisfies the following differential equation in y-coordinates:

$$\dot{y} = F(y,t) + \frac{\partial \psi}{\partial x}(\psi^{-1}(y)) \left[g_0(\varepsilon, x) + \sum_{k=1}^{m} g_k(\varepsilon, x)\alpha_k(x,t) \right] \Bigg|_{x=\psi^{-1}(y)} \tag{16}$$

with

$$F(y,t) := \frac{\partial \psi}{\partial x}(\psi^{-1}(y))\tilde{f}(\psi^{-1}(y),t).$$

By the arguments in the previous paragraph, the components of F verify $F_i = O(\|\cdot\|)$ $(i = 1, \ldots, n)$.

Since ψ maps e_q to itself, we have $\overline{\alpha} := \alpha_p(\psi^{-1}(\mathrm{e}_q), t_0) = \alpha_p(\mathrm{e}_q, t_0)$. By assumption, $\overline{\alpha} \neq 0$ and α_p is continuous. Thus, by continuity of ψ^{-1}, we infer the existence of reals $\varrho > 0$, $K_1 > 0$, and of an interval $\mathcal{T} := [t_0, t_0 + \tau]$, with $t_0 < t_0 + \tau < t_1$, such that for any $(y,t) \in B_\varrho(\mathrm{e}_q) \times \mathcal{T}$:

$$|\alpha_p(\psi^{-1}(y),t)| = \mathrm{sgn}(\overline{\alpha})\alpha_p(\psi^{-1}(y),t) > K_1. \tag{17}$$

Now, pick any integer $a > 0$ such that $2a > \delta_p + r_q > 0$ and choose $\eta > 0$ such that for any $y \in \mathbb{R}^n$, with $y_q = 1$, one has $\|y - \mathrm{e}_q\| \leq \varrho$ whenever $\|y - \mathrm{e}_q\|_{2a} \leq \eta$ (this can always be achieved since the $2a$-norm $\|\cdot\|_{2a}$ and the Euclidean 2-norm $\|\cdot\|$ are equivalent). Define

$$\mathcal{V}(y) = \frac{1}{2a}\left[\eta^{2a} y_q^{2a} - \sum_{i=1, i\neq q}^{n} y_i^{2a}\right], \tag{18}$$

and

$$\mathcal{A} \;\; := \;\; \{y \in \mathbb{R}^{*n} : y_q > 0, \;\; \eta^{2a} y_q^{2a} > \sum_{i=1, i\neq q}^{n} y_i^{2a}\}.$$

It is easy to check that $\mathcal{A}$ is an open set and that its boundary is given by

$$\partial\mathcal{A} \;\; = \;\; \{y \in \mathbb{R}^{*n} : y_q > 0, \;\; \eta^{2a} y_q^{2a} = \sum_{i=1, i\neq q}^{n} y_i^{2a}\} \cup \{0\}.$$

Evidently, $\mathcal{V}$ is strictly positive in $\mathcal{A}$, whereas it vanishes in $\partial\mathcal{A}$. It remains to prove that the assumption (12) of Proposition 1 is also verified.

In view of the choice of η, the inequality in (17) holds for $t \in \mathcal{T}$ and y in the intersection of $\mathcal{A}$ with the hyper-plane determined by $y_q = 1$. Furthermore, the mapping $(y,t) \mapsto \alpha_p(\psi^{-1}(y),t)$ is homogeneous of degree $-\delta_p$, so

$$\forall (y,t) \in \mathcal{A} \times \mathcal{T} : \quad |\alpha_p(\psi^{-1}(y),t)| > 0. \tag{19}$$

Denote by $\mathcal{F}$ the vector field on the right-hand member of (16). By direct differentiation, and using (7) and (19), we see that for every $(y,t) \in \mathcal{A} \times \mathcal{T}$:

$$(L_{\mathcal{F}}\mathcal{V})(y,t) = \frac{\partial \mathcal{V}}{\partial y}(y)\left[F(y,t) + \frac{\partial \psi}{\partial x}(\psi^{-1}(y))\varepsilon|\alpha_p(\psi^{-1}(y),t)|\mathrm{e}_q\right].$$

The (i,j)-th entry of the required Jacobian matrix $(\partial\psi/\partial x)(x)$ is given by:

$$\frac{\partial \psi_i}{\partial x_j}(\psi^{-1}(y)) = \rho^{1-r_i}(x)\left[\frac{1-r_i}{r_j}\rho^{-2c}(x)x_i x_j^{\frac{2c}{r_j}-1} + \delta_{ij}\right]\Bigg|_{x=\psi^{-1}(y)}, \tag{20}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. $(L_{\mathcal{F}}\mathcal{V})(y,t)$ can also be written as

$$(L_{\mathcal{F}}\mathcal{V})(y,t) = \mathcal{V}_1(y,t) + \mathcal{V}_2(y,t),$$

with

$$\begin{aligned}
\mathcal{V}_1(y,t) &:= \sum_{i=1}^{n} \frac{\partial \mathcal{V}}{\partial y_i}(y)F_i(y,t),\\
\mathcal{V}_2(y,t) &:= \varepsilon|\alpha_p(\psi^{-1}(y),t)|[\xi(y)+\zeta(y)],
\end{aligned}$$

and

$$\begin{aligned}
\xi(y) &:= \frac{\partial \mathcal{V}}{\partial y_q}(y)\cdot\frac{\partial \psi_q}{\partial x_q}(\psi^{-1}(y))\\
&= \eta^{2a}y_q^{2a-1}\rho^{1-r_q}(x)\left[\frac{1-r_q}{r_q}\left(\frac{x_q^{\frac{1}{r_q}}}{\rho(x)}\right)^{2c}+1\right]\Bigg|_{x=\psi^{-1}(y)}, \qquad (21)\\
\zeta(y) &:= \sum_{i=1,i\neq q}^{n}\frac{\partial \mathcal{V}}{\partial y_i}(y)\cdot\frac{\partial \psi_i}{\partial x_q}(\psi^{-1}(y))\\
&= -\sum_{i=1,i\neq q}^{n} y_i^{2a-1}\rho^{1-r_i}(x)\left[\frac{1-r_i}{r_q}\left(\frac{x_q^{\frac{1}{r_q}}}{\rho(x)}\right)^{2c}\frac{x_i}{x_q}\right]\Bigg|_{x=\psi^{-1}(y)}. \qquad (22)
\end{aligned}$$

Note that $\partial\mathcal{V}/\partial y_i$ $(i = 1,\ldots,n)$ is homogeneous of degree $2a-1$, thus $\mathcal{V}_1 = O(\|\cdot\|^{2a})$. Furthermore, since

$$\frac{|1-r_q|}{r_q} < 1 \qquad \text{and} \qquad \left(\frac{x^{\frac{1}{r_q}}}{\rho(x)}\right)\Bigg|_{x=\psi^{-1}(y)} \leq 1 \quad (\forall y \neq 0),$$

the term within brackets in (21), and hence $\xi(y)$, are strictly positive. On the other hand, for $i = 1,\dots,n$, we have $r_i \geq 1$, so $1 - r_i = -|r_i - 1|$. Therefore, whenever $y \neq 0$

$$\zeta(y) = \sum_{i=1, i\neq q}^{n} \operatorname{sgn}(y_i) \frac{\operatorname{sgn}(x_i)}{\operatorname{sgn}(x_q)} |y_i|^{2a-1} \rho^{1-r_i}(x) \left(\frac{|r_i - 1|}{r_q}\right) \left(\frac{x_q^{\frac{1}{r_q}}}{\rho(x)}\right)^{2c} \left|\frac{x_i}{x_q}\right| \Bigg|_{x=\psi^{-1}(y)} .$$

But ψ preserves the sign component-wise, i.e. $\operatorname{sgn}(y_i) = \operatorname{sgn}(\psi_i(x)) = \operatorname{sgn}(x_i)$, thus $\zeta(y) \geq 0$. From (19), and since $\xi(y) > 0$ and $\zeta(y) > 0$, we conclude that $\mathcal{V}_2(y,t) > 0$ for every $(y,t) \in \mathcal{A} \times \mathcal{T}$. By inspection of (21) and (22), we further deduce that ξ and ζ are homogeneous of degree $2a - r_q$. In consequence, $\mathcal{V}_2$ is homogeneous of degree

$$d := 2a - \delta_p - r_q > 0.$$

Let $S^{n-1} := \{y \in \mathbb{R}^n : \|y\| = 1\}$ and $\overline{\mathcal{A}} := \mathcal{A} \cup \partial\mathcal{A}$. Using the homogeneity of $\mathcal{V}_2$, for any $(y,t) \in \mathcal{A} \times \mathcal{T}$

$$\frac{\mathcal{V}_2(y,t)}{\|y\|^d} = \varepsilon|\alpha_p(\psi^{-1}(\overline{y}),t)|[\xi(\overline{y}) + \zeta(\overline{y})],$$

with $\overline{y} := y/\|y\| \in S^{n-1} \cap \mathcal{A}$. One easily verifies that $(S^{n-1} \cap \mathcal{A}) \subset B_\varrho(\mathrm{e}_q)$, so (17) holds and $|\alpha_p(\psi^{-1}(\overline{y}),t)| \geq K_1$. Also, ξ and ζ are continuous away from the origin, hence they attain their minima on the compact set $S^{n-1} \cap \overline{\mathcal{A}}$. These minima are strictly positive, as follows from the homogeneity of ξ and ζ and from the fact that both functions are strictly positive in $\mathcal{A}$. One can thus define

$$K_2 := \varepsilon K_1 \min_{\overline{y} \in S^{n-1} \cap \overline{\mathcal{A}}} \{\xi(\overline{y}) + \zeta(\overline{y})\}$$

to conclude that for every $(y,t) \in \mathcal{A} \times \mathcal{T}$

$$\frac{\mathcal{V}_2(y,t)}{\|y\|^d} \geq K_2 > 0, \tag{23}$$

and

$$\begin{aligned} (L_{\mathcal{F}}\mathcal{V})(y,t) &= \|y\|^d \left(\|y\|^{2a-d} \frac{\mathcal{V}_1(y,t)}{\|y\|^{2a}} + \frac{\mathcal{V}_2(y,t)}{\|y\|^d}\right) \\ &\geq \|y\|^d \left(\|y\|^{2a-d} \frac{\mathcal{V}_1(y,t)}{\|y\|^{2a}} + K_2\right). \end{aligned} \tag{24}$$

In addition to being $O(\|\cdot\|^{2a})$, $\mathcal{V}_1$ is the sum of homogeneous functions of degree $2a$ and higher order terms (this can be seen from its definition and from the fact that F is the sum of a homogeneous vector field of degree 0 and higher order vector fields). It follows that, by virtue of $2a - d > 0$, the term

$$\mathcal{Q}(y,t) := \|y\|^{2a-d} \frac{|\mathcal{V}_1(y,t)|}{\|y\|^{2a}} \tag{25}$$

tends to zero as y does, uniformly for t in $\mathcal{T}$. Then, there exist $\kappa' > 0$ and $K_3 \in \,]0, K_2[$ such that

$$\forall (y,t) \in [\mathcal{A} \cap B_{\kappa'}(0)] \times \mathcal{T} : \quad \mathcal{Q}(y,t) < K_3 \quad \text{and} \quad (L_{\mathcal{F}}\mathcal{V})(y,t) > 0.$$

Now, pick $(y,t) \in [\mathcal{A} \cap B_{\kappa'}(0)] \times \mathcal{T}$ arbitrarily. Using the inequalities $\mathcal{V}(y) > 0$ and $\|y\|^{2a} \geq y_q^{2a}$, and also the definition (18) of $\mathcal{V}$, we get $\|y\| \geq (2a/\eta^{2a})^{\frac{1}{2a}} [\mathcal{V}(y)]^{\frac{1}{2a}} > 0$ and, since $d > 0$,

$$\|y\|^d \geq \left(\frac{2a}{\eta^{2a}}\right)^{\frac{d}{2a}} [\mathcal{V}(y)]^{\frac{d}{2a}} > 0.$$

This last inequality, combined with (24), whose right-hand member is strictly positive, yields

$$(L_{\mathcal{F}}\mathcal{V})(y,t) \geq \left(\frac{2a}{\eta^{2a}}\right)^{\frac{d}{2a}} \left(\|y\|^{2a-d} \frac{\mathcal{V}_1(y,t)}{\|y\|^{2a}} + K_2\right) [\mathcal{V}(y)]^{\frac{d}{2a}}.$$

Setting

$$K := \left(\frac{2a}{\eta^{2a}}\right)^{\frac{d}{2a}} (K_2 - K_3) > 0,$$

we finally obtain

$$\forall (y,t) \in [\mathcal{A} \cap B_{\kappa'}(0)] \times \mathcal{T} : \quad (L_{\mathcal{F}}\mathcal{V})(y,t) \geq K [\mathcal{V}(y)]^{\frac{d}{2a}}.$$

In order to return to x-coordinates, let

$$V(x) := \mathcal{V}(\psi(x)), \quad \text{and} \quad A := \psi^{-1}(\mathcal{A}),$$

and let f denote the vector field in the right-hand member of (14). It is straightforward to check that the real number

$$\kappa := \min_{y \in \overline{\mathcal{A}},\ \|y\| = \kappa'} \|\psi^{-1}(y)\|$$

exists, and that $(L_f V)(x,t) \geq K[V(x)]^{\frac{d}{2a}}$ for every $(x,t) \in [A \cup B_\kappa(0)] \times \mathcal{T}$. Since $\gamma := \frac{d}{2a} \in \,]0,1[$, the assumption (12) of Proposition 1 is thus verified and the proof is complete. ∎

4.2.2 Proof of Theorem 2

Let

$$\dot{x} = \sum_{k=1}^{m} b_k(x) u_k \tag{26}$$

be any r-homogeneous approximation of system (9) with coordinate change $x = \varphi(z)$. Also, consider any continuous homogeneous ρ-exponential stabilizer α for (9) meeting the requirements of Definition 3. From the previous definitions, the vector fields b_k are r-homogeneous of degree $\delta_k < 0$, whereas the feedback functions α_k are r-homogeneous of degree $-\delta_k$ $(k = 1, \dots, m)$. Let $S := \{b_1 + h_1, \dots, b_m + h_m\}$ and $S_A := \{b_1, \dots, b_m\}$ denote the sets of vector fields (expressed in x-coordinates) which define the nominal system and homogeneous approximation (26), respectively. Let $\mathrm{Lie}(S \backslash \{b_p + h_p\})$ be the Lie algebra spanned by the elements of S after removing the p-th vector field $b_p + h_p$, and define $\mathrm{Lie}(S_A \backslash \{b_p\})$ accordingly. Given a set $\mathcal{X}$ of vector fields, we let $\mathcal{X}(0) := \mathrm{span}\{X(0) : X \in \mathcal{X}\}$.

The following technical lemma states that if the "rank-loss" property (10) holds for the nominal system, then a similar property holds for the approximation (26).

Lemma 1 *Under the assumptions of Theorem 2, one has*

$$\forall p \in \{1, \dots, m\} : \quad \dim \mathrm{Lie}(S_A \backslash \{b_p\})(0) < n - 1. \tag{27}$$

(*Proof in the Appendix*).

We claim that the weights r_i are not all equal, so $N \geq 2$. To prove this, suppose that $r = (1, \dots, 1)$. It follows from P5 that $b_1, \dots, b_m$ are constant so that any bracket involving one or more of these vector fields is identically zero. But the approximation (26) satisfies the LARC, so the family $(b_1(0), \dots, b_m(0))$ spans $\mathbb{R}^n$ and $m \geq n$. Therefore, if we remove any b_k, the accessibility distribution of the approximation loses at most one dimension at the origin. In other words $\dim \mathrm{Lie}(S_A \backslash \{b_k\})(0) \geq n - 1$, which contradicts (27).

Now let $\mathbf{x}$ be the r-partition of the coordinate vector x. Using properties P3 and P6, we can write the approximation (26) in the form

$$\begin{array}{rcl} \dot{\mathbf{x}}_1 & = & F_1 u \\ \dot{\mathbf{x}}_2 & = & F_2(\mathbf{x}_1) u \\ & \vdots & \\ \dot{\mathbf{x}}_N & = & F_N(\mathbf{x}_1, \dots, \mathbf{x}_{N-1}) u \end{array}$$

with F_1 representing a constant matrix, $F_2, \dots, F_N$ matrix-valued functions, and $u = (u_1, \dots, u_m)^T$ the control vector.

We shall now prove that $F_N(0) = 0$ by contradiction. Suppose that $F_N(0) \neq 0$. Then there exists at least one vector field $b_j \in S_A$ such that $\mathbf{b}_{j,N}(0)$ is nonzero (recall that $\mathbf{b}_j$ denotes the r-partition of b_j). The components of $\mathbf{b}_{j,N}$ are thus r-homogeneous of degree 0 and b_j is r-homogeneous of degree $\delta_j = -r_n = -r^N$. From P3 we infer that $\mathbf{b}_{j,1}, \dots, \mathbf{b}_{j,N-1}$ are identically zero, so b_j is a constant vector field. Since every vector field in $\mathrm{Lie}(S_A)$ is r-homogeneous of negative degree, it follows from P2 and P4 that any bracket containing b_j is zero. Hence, removing b_j amounts to removing only one vector from $\mathrm{Lie}(S_A)(0)$, so $\dim \mathrm{Lie}(S_A \backslash \{b_j\})(0) \geq n - 1$, in contradiction with (27).

Finally, we show that the nominal system verifies the assumptions of Theorem 1. For this it suffices to prove that the feedback α is such that for some $p \in \{1, \dots, m\}$ and some

$t_0 \in \mathbb{R}$

$$\alpha_p(\mathrm{e}_n, t_0) \neq 0 \quad \text{and} \quad -\delta_p < r_n = r^N, \tag{28}$$

with $\mathrm{e}_n = (0, \ldots, 0, 1)^T \in \mathbb{R}^n$. Let us again proceed by contradiction and assume that $\alpha_j(\mathrm{e}_n, t) = 0$ for all $j \in \{1, \ldots, m\}$ and all $t \in \mathbb{R}$. It follows that $\dot{\mathbf{x}} = 0$ at (e_n, t) for any $t \in \mathbb{R}$. This implies in turn, using the homogeneity of α_j, that the set $\{\Delta^r_\lambda(\mathrm{e}_n) : \lambda > 0\} \subset \mathbb{R}^n$ (the orbit of e_q under the action of Δ^r) is an equilibrium manifold for the closed-loop system. Since this set intersects non-trivially with every neighborhood of $x = 0$, this contradicts the fact that the latter is asymptotically stable. Now suppose that $-\delta_p = r_n$ (the case $-\delta_p > r_n$ is ruled out by P4). Then, in view of P3, all components of $\mathbf{b}_{p,N}$ are constant. But if $F_N(0) = 0$, then $\mathbf{b}_{p,N} = 0$ and hence $b_p \equiv 0$, which again leads to a contradiction with (27). Therefore (28) holds, and the conclusion of Theorem 2 follows by application of Theorem 1 with $f_0 \equiv 0$ and $q = n$. ∎

5 Examples

In this section we discuss two simple examples which illustrate how the previous theorems can be used to assert the non-robustness of feedback stabilizers.

Example 1. (*ρ-Exponential stabilization of the attitude of a rigid spacecraft with two controls*). Although for these systems no given (fixed) attitude can be stabilized by means of continuous pure-state feedback —since they do not meet Brockett's condition— continuous *time-periodic* ρ-exponential stabilizers have been proposed in Coron and Kerai [5] and in Morin and Samson [21]. Let us prove that the ρ-exponential stabilizers given in [21] are not robust by showing that the assumptions of Theorem 1 are verified. The nominal model of this system (cf. [21] and the references therein for the physical meaning of variables) can be written in the form (1) with

$$f_0(z) = \frac{1}{2} \begin{bmatrix} z_4 + z_2 z_6 - z_3 z_5 + z_1 \sum_{i=1}^3 z_i z_{i+3} \\ z_5 + z_3 z_4 - z_1 z_6 + z_2 \sum_{i=1}^3 z_i z_{i+3} \\ z_6 + z_1 z_5 - z_2 z_4 + z_3 \sum_{i=1}^3 z_i z_{i+3} \\ 2a z_5 z_6 \\ 2b z_4 z_6 \\ 2c z_4 z_5 \end{bmatrix},$$

$$f_1 = (0,0,0,1,0,0)^T, \quad f_2 = (0,0,0,0,1,0)^T.$$

and a, b, c real constants. A dilation of weight $r = (1,1,2,1,1,2)$ is considered, and it can be immediately verified that $\dot{x} = b_0(x) + b_1 u_1 + b_2 u_2$, with

$$b_0(x) = \tfrac{1}{2}(x_4, x_5, x_6, 0, 0, 2c x_4 x_5)^T,$$
$$b_1 = f_1, \quad b_2 = f_2,$$

is an r-homogeneous approximation with trivial change of coordinates $x = z$. The vector fields b_0, b_1, b_2 are r-homogeneous of degrees $\delta_0 = 0$, $\delta_1 = -1$ and $\delta_2 = -1$, respectively. The control law given in [21] is:

$$\begin{aligned} \alpha_1(x,t) &= -k_3\left[x_4 + k_1 x_1 + \overline{\rho}(x)\sin(t/\varepsilon)\right], \\ \alpha_2(x,t) &= -k_4\left[x_5 + k_2 x_2 - \frac{1}{\overline{\rho}(x)}(x_3 + x_6)\sin(t/\varepsilon)\right], \end{aligned}$$

where $\{k_i\}_{i=1,\dots,4}$ and ε are positive reals conveniently chosen, and $\overline{\rho} = \rho \circ \sigma : \mathbb{R}^n \to \mathbb{R}^n$ is the composition of any r-norm ρ with $\sigma : x \mapsto (x_1, x_2, x_3, 0, 0, x_6)^T$. One readily checks that these control functions are both r-homogeneous of degree $-\delta_1 = -\delta_2 = 1$. Now, picking any $t_0 \in \mathbb{R}$ with $(t_0/\varepsilon) \not\equiv 0 \mod \pi$, we have $\alpha_2(\mathrm{e}_6, t_0) = -\sin(t_0/\varepsilon)/\overline{\rho}(\mathrm{e}_6) \neq 0$. But $-\delta_2 = 1 < 2 = r_6$, therefore the assumptions of Theorem 1 are verified for $p = 2$ and $q = 6$.

Example 2.(*ρ-Exponential stabilization of chained-form systems*). As an application of Theorem 2, consider a chained-form system

$$\dot{x} = b_1(x)u_1 + b_2 u_2 \quad (n \geq 3), \tag{29}$$

with $b_1(x) = (1, x_3, \dots, x_n, 0)^T$ and $b_2 = (0, \dots, 0, 1)^T$. This canonical form can be used to model the kinematics of some nonholonomic systems, including several mobile robots (see for example Murray [23], Sørdalen [33], Samson [30]). In this case, (29) is a homogeneous system, where b_1, b_2 are r-homogeneous of degrees -1 and $-a < 0$ respectively, and $r = (1, a+n-2, a+n-3, \dots, a)$. One readily verifies that $\mathrm{Lie}(b_1, b_2)$ is nilpotent, and that the accessibility distribution spans the whole tangent space so the system is locally controllable. By removing either b_1 or b_2, the condition (10) of Theorem 2 holds trivially (for in this case the Lie algebra contains only linear combinations of a single vector field, hence the dimension of the accessibility distribution is at most $1 < n-1$). Therefore, any static continuous time-varying feedback law which asymptotically stabilizes the origin of (29) and which renders the closed-loop system r-homogeneous of degree zero, is not robust in the sense of Definition 2. The same conclusion holds if the homogeneous ρ-exponential stabilizer is constructed from a chained-form homogeneous approximation of the original system. Note that such an approximating system may exist even when the original system cannot itself be converted into the chained form. This is illustrated in Lizárraga et al. [14], where the authors propose time-periodic feedbacks which asymptotically stabilize certain configurations of the general N-trailer system with off-axle hitching. By application of Theorem 2, one concludes that those feedbacks are not robust either.

6 Concluding Remarks

We have pointed out sufficient conditions under which the origin of an affine control system, ρ-exponentially stabilized by a continuous homogeneous feedback, may become unstable in the presence of arbitrarily small perturbations in the control vector fields. Weaker non-robustness conditions may possibly be obtained, and theorems 1 and 2 might shed some light on methods for proving new results in this direction. For instance, it seems reasonable to expect that the non-robustness results hereby presented extend to the more general situation in which Hölder continuous (but not necessarily homogeneous) time-varying feedback is used to exponentially stabilize a system with non-stabilizable linear approximation.

Interestingly, the results in this paper may be seen as a partial theoretical justification for the exploration of feedback laws (such as those proposed by Bennani and Rouchon [2, 1], or the ones recently introduced by Morin and Samson [22]) which preserve a continuous dependence on the state, but are only updated at adequate discrete time-instants.

Theorem 1 points out admissible perturbations that yield instability, but it obviously fails to characterize all of them. It is also clear that not every admissible perturbation produces instability, which is a direct consequence of the existence of homogeneous Lyapunov functions for the nominal closed-loop system. Therefore, a possible direction of future research might consist in classifying *the classes of perturbations with respect to which a given type of feedback is (or is not) robust.* Moreover, one should keep in mind that only instability in the sense of Lyapunov has been considered; the fact that a feedback is not robust in the sense of Definition 2 does not imply that it does not exhibit more "lenient" stabilization properties. For instance, the system may well be Lyapunov-unstable and still be *practically stable* in the sense discussed e.g. in Hahn [9, Ch. IV].

7 Appendix

Proof of Lemma 1. We need to show that for all $p \in \{1, \dots, m\}$

$$\dim \mathrm{Lie}(S \backslash \{b_p + h_p\})(0) < n-1 \quad \Longrightarrow \quad \dim \mathrm{Lie}(S_A \backslash \{b_p\})(0) < n-1,$$

for which it suffices to prove

$$\forall p \in \{1, \dots, m\}: \quad \dim \mathrm{Lie}(S_A \backslash \{b_p\})(0) \leq \dim \mathrm{Lie}(S \backslash \{b_p + h_p\})(0).$$

Choose any $p \in \{1, \dots, m\}$ and let $\mathcal{B}_A := \bigcup_{\ell=1}^{\infty} C_\ell$, where the sets $\{C_\ell\}_{\ell \geq 1}$ are inductively defined by

$$\begin{aligned} C_1 &:= S_A \backslash \{b_p\} \\ C_\ell &:= [C_1, C_{\ell-1}] \backslash \{0\} \qquad (\ell > 1). \end{aligned}$$

It is easy to show (see e.g. Nijmeijer and van der Schaft [24, Chap. 3]) that any vector field in $\mathrm{Lie}(S_A \backslash \{b_p\})$ can be written as a linear combination (with real coefficients) of elements of $\mathcal{B}_A$. Moreover, by property P2, we see that every element of $\mathcal{B}_A$ is r-homogeneous of strictly negative degree. Let us now define the mapping $\beta : \mathcal{B}_A \to \mathrm{Lie}(S \backslash \{b_p + h_p\})$ given by

$$\beta(X) := \begin{cases} b_k + h_k & \text{if } X = b_k \in C_1, \\ [b_k + h_k, \beta(X')] & \text{if } X = [b_k, X'] \in C_\ell \text{ with } X' \in C_{\ell-1}. \end{cases}$$

We claim that for any $\ell \geq 1$ and any vector field $X \in C_\ell$, r-homogeneous of degree $\delta < 0$, the image of X by β has the form

$$\beta(X) = X + \xi,$$

with $\xi_i = o(\rho^{r_i+\delta})$ $(i = 1, \dots, n)$. Let us proceed by induction. For $\ell = 1$ (i.e. if $X = b_k \in C_1$) we have $\beta(b_k) = b_k + h_k$, so the claim is obvious by the definition of r-homogeneous approximation. Now suppose that the claim holds for $\ell = w \geq 1$ and choose $b_k \in C_1$ and $X' \in C_w$ such that $X := [b_k, X'] \in C_{w+1}$. The vector fields b_k and X' are r-homogeneous of degrees $\delta_k < 0$ and $\delta' < 0$ respectively, so it follows from property P2 that X is r-homogeneous of degree $\delta_k + \delta'$. Moreover, by the induction assumption, $\beta(X') = X' + \xi'$ with $\xi'_i = o(\rho^{r_i+\delta'})$ $(i = 1, \dots, n)$, so using the bilinearity of the Lie bracket we get

$$\begin{aligned} \beta(X) &= [b_k + h_k, \beta(X')] \\ &= [b_k + h_k, X' + \xi'] \\ &= [b_k, X'] + [b_k, \xi'] + [h_k, X'] + [h_k, \xi']. \end{aligned} \tag{30}$$

Since the first term on the right of (30) is equal to X, we only need to show that each component of the remaining three terms is $o(\rho^{r_i+\delta_k+\delta'})$. With this in mind, we compute the i-th component of $[b_k, \xi'](x)$

$$[b_k, \xi']_i(x) = \sum_{j=1}^{n} \frac{\partial \xi'_i}{\partial x_j} b_{k,j}(x) - \sum_{j=1}^{n} \frac{\partial b_{k,i}}{\partial x_j} \xi'_j(x). \tag{31}$$

Each $\partial b_{k,i}/\partial x_j$ is r-homogeneous of degree $r_i - r_j + \delta_k$ and $\xi'_j = o(\rho^{r_j+\delta'})$, so the second sum in (31) is $o(\rho^{r_i+\delta_k+\delta'})$. Furthermore, it is straightforward to prove, by means of a Taylor expansion with integral remainder, that $\partial \xi'_i/\partial x_j = o(\rho^{r_i-r_j+\delta'})$. But $b_{k,j}$ is r-homogeneous of degree $r_j + \delta_k$, so the first sum is also $o(\rho^{r_i+\delta_k+\delta'})$, as required. Similar arguments apply to the components of $[h_k, X']$ and $[h_k, \xi']$, so that $\beta(X)$ is as claimed.

Next, pick any $X \in \mathcal{B}_A$ such that $X(0) \neq 0$ and let $Y := \beta(X) = X + \xi$. Denote their r-partitions as $\mathbf{X}$ and $\mathbf{Y} = \mathbf{X} + \boldsymbol{\xi}$, respectively. Since X is r-homogeneous of degree $\delta < 0$, it follows that the components in $\mathbf{X}_j$ are r-homogeneous of degree $r^j + \delta$, and those in $\boldsymbol{\xi}_j$ are $o(\rho^{r^j+\delta})$ $(j = 1, \dots, N)$. Using the fact that $X(0) \neq 0$, together with property P3, we deduce the existence of an integer J such that $r^J = \delta$ and such that *all* the nonzero components of $\mathbf{X}(0)$ are in $\mathbf{X}_J(0)$. Besides, for any j such that $r^j \geq -\delta$, the relation

$$\lim_{\rho(x)\to 0} \frac{|\xi_i(x)|}{\rho^{r^j+\delta}(x)} = 0$$

and the continuity of ξ imply that $\boldsymbol{\xi}_j(0) = 0$. Therefore, $\mathbf{X}(0)$ and $\boldsymbol{\xi}(0)$ are in the form

$$\mathbf{X}(0) = (0, \dots, 0, K_J, 0, \dots, 0)^T, \quad \boldsymbol{\xi}(0) = (K_1, \dots, K_{J-1}, 0, \dots, 0)^T, \tag{32}$$

with $K_1, \ldots, K_J$ constant sub-vectors, and K_J located at the J-th entry in the partition of $\mathbf{X}(0)$. Let $d := \dim \mathrm{Lie}(S_A \backslash \{b_p\})(0)$. Since $\mathcal{B}_A$ spans $\mathrm{Lie}(S_A \backslash \{b_p\})$, we can choose a family $\{X_k\}_{1 \leq k \leq d}$ of vector fields from $\mathcal{B}_A$ such that the vectors $\{X_k(0)\}_{1 \leq k \leq d}$ are linearly independent. Under these conditions, one can construct a square block-diagonal full-rank matrix M_A, whose columns are taken from $\{X_k(0)\}_{1 \leq k \leq d}$, by eventually deleting some lines. Let $Y_k := \beta(X_k) = X_k + \xi_k$ $(k = 1, \ldots, d)$, and construct the matrix M by repeating the *same* procedure as for M_A, but this time using the corresponding vectors from $\{Y_k(0)\}_{1 \leq k \leq d}$. Given the structure (32), we see that the nonzero elements which are present in M and *not* in M_A are located above the block-wise diagonal, therefore the rank of M is also d. As a consequence, $\dim \mathrm{Lie}(S_A \backslash \{b_p\})(0) \leq \dim \mathrm{Lie}(S \backslash \{b_p + h_p\})(0)$, and since p was arbitrarily chosen, the conclusion follows. ∎

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